Unit - 2 : Inverse Laplace Transform

Formulae :-

1. $L^{-1}\left(\frac{1}{s}\right) = 1$

2. $L^{-1}\left(\frac{k}{s}\right) = k$

3. $L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

4. $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

5. $L^{-1}\left(\frac{a}{a^2+s^2}\right) = \sin at$

6. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

7. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

8. $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

9. $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$

10. $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$

First Shifting property :-

If $L^{-1}[\bar{F}(s)] = F(t)$

$$L^{-1}[\bar{F}(s-a)] = e^{at} F(t)$$
$$= e^{at} L^{-1}[\bar{F}(s)]$$

$$L^{-1}[\bar{F}(s+a)] = e^{-at} F(t)$$
$$= e^{-at} L^{-1}[\bar{F}(s)]$$

* Examples:

1. Find $L^{-1}\left[\dfrac{4s+15}{16s^2-25}\right]$

→ Consider

$$\frac{4s+15}{16s^2-25} = \frac{4s+15}{16\left[s^2-\frac{25}{16}\right]}$$

$$= \frac{4s+15}{16\left[s^2-\left(\frac{5}{4}\right)^2\right]}$$

$$L^{-1}\left[\frac{4s+15}{16s^2-25}\right] = \frac{1}{16} L^{-1}\left[\frac{4s}{s^2-(5/4)^2}\right] + \frac{15}{16} L^{-1}\left[\frac{1}{s^2-(5/4)^2}\right]$$

$$= \frac{4}{16} L^{-1}\left[\frac{s}{s^2-(5/4)^2}\right] + \frac{15}{16} L^{-1}\left[\frac{1}{s^2-(5/4)^2}\right]$$

$$= \frac{1}{4}\cosh\frac{5}{4}t + \frac{15}{16} \times \frac{1}{5/4} L^{-1}\left[\frac{5/4}{s^2-(5/4)^2}\right]$$

$$= \frac{1}{4}\cosh\frac{5}{4}t + \frac{15}{16}\times\frac{4}{5}\sinh\frac{5}{4}t$$

$$= \frac{1}{4}\cosh\frac{5}{4}t + \frac{3}{4}\sinh\frac{5}{4}t$$

2. $L^{-1}\left[\frac{s+1}{s^2+s+1}\right]$

→

Consider $\frac{s+1}{s^2+s+1}$ —— (I)

ax^2+bx+c

$a=1,\ b=1,\ c=1$

Complete square term $= \left(\frac{-b}{2a}\right)^2$

$$= \left(\frac{-1}{2\times1}\right)^2 = \left(\frac{-1}{2}\right)^2 = \frac{1}{4}$$

eq^n (I) becomes

$$\frac{s+1}{s^2+s+1} = \frac{s+1}{s^2+s+1/4+1-1/4}$$

$$= \frac{s+1}{(s+1/2)^2+\frac{4-1}{4}}$$

$$= \frac{s+1}{(s+1/2)^2+3/4}$$

$$\frac{s+1}{s^2+s+1} = \frac{s+1}{(s+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Taking L^{-1} on both

$$L^{-1}\left[\frac{s+1}{s^2+s+1}\right] = L^{-1}\left[\frac{s+1}{(s+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= L^{-1}\left[\frac{s+1/2+1/2}{(s+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$L^{-1}\left[\frac{s+1}{s^2+s+1}\right] = L^{-1}\left[\frac{s+1/2}{(s+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(s+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= e^{-1/2t}L^{-1}\left[\frac{s}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right] + \frac{1}{2}e^{-1/2t}L^{-1}\left[\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= e^{-1/2t}\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}e^{-1/2t}\frac{1}{\frac{\sqrt{3}}{2}}\sin\frac{\sqrt{3}}{2}t$$

$$= e^{-1/2t}\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}e^{-1/2t}\sin\frac{\sqrt{3}}{2}t$$

$$= e^{-1/2t}\left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]$$

3. $L^{-1}\left[\dfrac{15}{s^2+4s+13}\right]$

→

Consider,

$$\frac{15}{s^2+4s+13} \quad \text{—} \quad (I)$$

$$ax^2+bx+c$$

$$a=1, \quad b=4, \quad c=13$$

Complete square term $= \left(\frac{-b}{2a}\right)^2$

$$= \left(\frac{-4}{2\times1}\right)^2 = (-2)^2 = 4$$

eq^n (I) becomes

$$\frac{15}{s^2+4s+13} = \frac{15}{s^2+4s+4+13-4}$$

$$= \frac{15}{(s+2)^2+3^2}$$

$$\frac{15}{s^2+4s+13} = \frac{15}{(s+2)^2+3^2}$$

Taking L^{-1} on both side

$$L^{-1}\left[\frac{15}{s^2+4s+13}\right] = L^{-1}\left[\frac{15}{(s+2)^2+3^2}\right]$$

$$= 15\, L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]$$

$$= 15e^{-2t}\, L^{-1}\left[\frac{1}{s^2+3^2}\right]$$

$$= 15\, e^{-2t}\, \frac{1}{3} \sin 3t$$

$$\boxed{L^{-1}\left[\frac{15}{s^2+4s+13}\right] = 5e^{-2t} \sin 3t}$$

4. $L^{-1}\left[\dfrac{s+8}{s^2+4s+5}\right]$

$\rightarrow$

Consider,

$$\frac{s+8}{s^2+4s+5} \quad \text{—— (1)}$$

$ax^2 + bx + c$

$a = 1, \quad b = 4, \quad c = 5$

Complete Square term $= \left(\dfrac{-b}{2a}\right)^2$

$$= \left(\frac{-4}{2\times 1}\right)^2 = (-2)^2 = 4$$

eq^n (I) becomes

$$\frac{s+8}{s^2+4s+5} = \frac{s+8}{s^2+4s+4+5-4}$$

$$= \frac{s+8}{(s+2)^2+1}$$

$$\frac{s+8}{s^2+4s+5} = \frac{s+8}{(s+2)^2+1}$$

Taking L^{-1} on both sides

$$L^{-1}\left[\frac{s+8}{s^2+4s+5}\right] = L^{-1}\left[\frac{s+8}{(s+2)^2+1}\right]$$

$$= L^{-1}\left[\frac{(s+2)+6}{(s+2)^2+1}\right]$$

$$L^{-1}\left[\frac{s+8}{s^2+4s+5}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] + 6\,L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{s}{s^2+1}\right] + 6e^{-2t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-2t}\cos t + 6e^{-2t}\frac{1}{1}\sin t$$

$$= e^{-2t}\cos t + 6e^{-2t}\sin t$$

$$= e^{-2t}\left[\cos t + 6\sin t\right]$$

Imp

$*$ Inverse laplace Transform of derivative of $\bar{F}(s)$

$$L^{-1}\left[\frac{d}{ds}[\bar{F}(s)]\right] = -tF(t)$$

$$L^{-1}\left[\frac{d^2}{ds^2}[\bar{F}(s)]\right] = (-1)^2 t^2 F(t)$$

Examples

1 Find $L^{-1}\left[\cancel{\log}\ \frac{1}{s^2}\log\left(1+\frac{1}{s^2}\right)\right]$

→

Consider

$$\bar{F}(s) = \log\left[1+\frac{1}{s^2}\right]$$

$$= \log\left[\frac{s^2+1}{s^2}\right]$$

$$= \log[s^2+1] - \log[s^2]$$

diff. w.r.t. 's'

$$\frac{d}{ds}[\bar{F}(s)] = \frac{d}{ds}\left[\log(s^2+1) - \log(s^2)\right]$$

$$= \frac{1}{s^2+1}\frac{d}{ds}(s^2+1) - \frac{1}{s^2}\frac{d}{ds}(s^2)$$

$$= \frac{2s}{s^2+1} - \frac{2s}{s^2}$$

$$\frac{d}{ds}[\bar{F}(s)] = \frac{2s}{s^2+1} - \frac{2}{s}$$

taking L^{-1} on both

$$L^{-1}\left[\frac{d}{ds}[\bar{F}(s)]\right] = L^{-1}\left[\frac{2s}{s^2+1} - \frac{2}{s}\right]$$

$$-tF(t) = 2L^{-1}\left[\frac{s}{s^2+1}\right] - 2L^{-1}\left[\frac{1}{s}\right]$$

$$-tF(t) = 2\cos t - 2\times 1$$

$$tF(t) = 2 - 2\cos t$$

$$\boxed{F(t) = \frac{2(1-\cos t)}{t}}$$

2. Find $L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right]$

→ Sol^n, let $\bar{F}(s) = \log\left(\frac{s+a}{s+b}\right)$

$$= \log(s+a) - \log(s+b)$$

Diff. w.r.t. s

$$\frac{d}{ds}[\bar{F}(s)] = \frac{d}{ds}\left[\log(s+a) - \log(s+b)\right]$$

$$= \frac{1}{s+a}\times 1 - \frac{1}{s+b}\times 1$$

$$= \frac{1}{s+a} - \frac{1}{s+b}$$

Taking L^{-1} on both sides

$$L^{-1}\left[\frac{d}{ds}\bar{F}(s)\right] = e^{-at} - e^{-bt}$$

$$t\,F(t) = e^{-at} - e^{-bt}$$

$$\boxed{F(t) = \frac{e^{-at} - e^{-bt}}{t}}$$

3) Find $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

→

Let $\bar{F}(s) = \log\left(\frac{s+1}{s-1}\right)$

$$\bar{F}(s) = \log(s+1) - \log(s-1)$$

Diff. w.r.t. s

$$\frac{d}{ds}[\bar{F}(s)] = \frac{d}{ds}\left[\log(s+1) - \log(s-1)\right]$$

$$\cancel{tF(t)} = \frac{1}{s+1} - \frac{1}{s-1}$$

Taking L^{-1} on both sides

$$L^{-1}\left[\frac{d}{ds}\left(\bar{F}(s)\right)\right] = L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$tF(t) = e^{-t} - e^{t}$$

$$\boxed{\cancel{t}F(t) = \frac{e^{t} - e^{t}}{t}}$$

Imp

4. Find $L^{-1}\left[\cot^{-1}\frac{(s+3)}{2}\right]$

$\rightarrow$ Let $\bar{F}(s) = \cot^{-1}\left(\frac{s+3}{2}\right)$

diff w.r.t s

$$\frac{d}{ds}[\bar{F}(s)] = \frac{d}{ds}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right]$$

use $\frac{d}{dx}[\cot^{-1}(x)] = \frac{-1}{1+x^2} \cdot \frac{d}{dx}(x)$

$$= \frac{-1}{1+\left(\frac{s+3}{2}\right)^2} \frac{d}{ds}\left(\frac{s+3}{2}\right)$$

$$= \frac{-1}{1+\frac{(s+3)^2}{4}} \times \frac{1}{2}(1+0)$$

$$= \frac{-1}{\frac{4+(s+3)^2}{4}} \times \frac{1}{2}$$

$$= \frac{-4^2}{4+(s+3)^2} \times \frac{1}{2}$$

$$= \frac{-2}{4+(s+3)^2}$$

Taking L^{-1} on both sides

$$L^{-1}\left[\frac{d}{ds}[\bar{F}(s)]\right] = L^{-1}\left[\frac{-2}{4+(s+3)^2}\right]$$

$$-tF(t) = -2\,L^{-1}\left[\frac{1}{4+(s+3)^2}\right]$$

$$-tF(t) = -2e^{-3t}\,L^{-1}\left[\frac{1}{4+s^2}\right]$$

$$-tF(t) = -2e^{-3t}\,\frac{1}{2}\sin 2t$$

$$\boxed{F(t) = \frac{e^{-3t}\sin 2t}{t}}$$

H.W

5. Find $L^{-1}\left[\cot^{-1}\left(\frac{s}{2}\right)\right]$

→ Let $\bar{F}(s) = \cot^{-1}\left(\frac{s}{2}\right)$

diff. w.r.t. s

$$\frac{d}{ds}[\bar{F}(s)] = \frac{d}{ds}\left[\cot^{-1}\left(\frac{s}{2}\right)\right]$$

use $\frac{d}{dx}[\cot^{-1}x] = \frac{-1}{1+x^2}$

$$= \frac{-1}{1+\left(\frac{s}{2}\right)^2} \cdot \frac{d}{ds}\left(\frac{s}{2}\right)$$

$$= \frac{-1}{1+\frac{s^2}{4}} \cdot \frac{1}{2}$$

$$= \frac{-1}{\frac{4+s^2}{4}} \times \frac{1}{2}$$

$$= \frac{-4}{4+s^2} \times \frac{1}{2}$$

$$= \frac{-2}{4+s^2}$$

Taking L^{-1} on both sides

$$L^{-1}\left[\frac{d}{ds}[\bar{F}(s)]\right] = L^{-1}\left[\frac{-2}{4+s^2}\right]$$

$$-tF(t) = -2\,L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$-tF(t) = -2 \times \frac{1}{2}\sin 2t$$

$$-tF(t) = -\sin 2t$$

$$\boxed{F(t) = \frac{\sin 2t}{t}}$$

6. Find $L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right]$

→

Let $\bar{F}(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$

Diff. w.r.t. s.

$$\frac{d}{ds}[\bar{F}(s)] = \frac{d}{ds}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right]$$

$$= \frac{1}{1+\left(\frac{2}{s^2}\right)^2}\,\frac{d}{ds}\left(\frac{2}{s^2}\right)$$

$$= \frac{1}{1+\frac{4}{s^4}}\left(\frac{2(-2)}{s^3}\right)$$

$$= \frac{1}{\frac{s^4+4}{s^4}} \times \frac{-4}{s^3}$$

$$= \frac{s^4}{s^4+4} \times \frac{-4}{s^3}$$

$$= \frac{-4s}{s^4+4}$$

$$\frac{d}{ds}[F(s)] = \frac{-4s}{s^4+4}$$

$$\frac{d}{ds}[F(s)] = \frac{1}{(s+1)^2+1} - \frac{1}{(s-1)^2+1}$$

Taking L^{-1} on both sides

$$L^{-1}\left[\frac{d}{ds}[F(s)]\right] = L^{-1}\left[\frac{1}{(s+1)^2+1} - \frac{1}{(s-1)^2+1}\right]$$

$$= L^{-1}\left[\frac{1}{(s+1)^2+1}\right] - L^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$= e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right] - e^{t} L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$-tF(t) = e^{-t}\sin t - e^{t}\sin t$$

$$tF(t) = e^{t}\sin t - e^{-t}\sin t$$

$$= \sin t\,[e^{t} - e^{-t}]$$

$$tF(t) = 2\sin t\left(\frac{e^{t}-e^{-t}}{2}\right)$$

$$tF(t) = 2\sin t \cdot \sinh t$$

$$\therefore \boxed{F(t) = \frac{2\sin t\,\sinh t}{t}}$$

Imp

* Partial Fraction.

1. $$\frac{ax^2+bx+c}{(x-1)(x+2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x-3)}$$

2. $$\frac{ax^2+bx+c}{(x^2+2)(x+3)} = \frac{Ax+B}{(x^2+2)} + \frac{C}{(x+3)}$$

3. $$\frac{ax^2+bx+c}{(x^2+2x+1)(x+4)} = \frac{Ax+B}{(x^2+2x+1)} + \frac{C}{(x+4)}$$

4. $$\frac{ax^2+bx+c}{(x+1)^3} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

Examples :-

1. Find $L^{-1}\left[\frac{s^2+2s-3}{s(s-3)(s+2)}\right]$

Consider

$$\frac{s^2+2s-3}{s(s-3)(s+2)} = \frac{A}{s} + \frac{B}{(s-3)} + \frac{C}{(s+2)} \quad \text{—} \ (2)$$

$$\frac{s^2+2s-3}{s(s-3)(s+2)} = \frac{A(s-3)(s+2) + B\cdot s(s+2) + C\cdot s(s-3)}{s(s-3)(s+2)}$$

$$s^2+2s-3 = A(s-3)(s+2) + B\,s(s+2) + C\cdot s(s-3)$$

$$s^2+2s-3 = A(s^2+2s-3s-6) + B(s^2+2s) + C\cdot(s^2-3s)$$

$$s^2+2s-3 = A(s^2-s-6) + B(s^2+2s) + C(s^2-3s)$$

$$s^2+2s-3 = As^2 - As - 6A + Bs^2 + B2s + Cs^2 - 3Cs$$
$$s^2+2s-3 = s^2(A+B+C) + s(-A+2B-3C) - 6A$$

Equatin powers s

$$A+B+C=1$$
$$-A+2B-3C=2$$
$$-6A=-3$$
$$\therefore \boxed{A=\frac{1}{2}}$$

$$B+C = 1-A$$
$$B+C = 1-\frac{1}{2}$$
$$\boxed{B+C=\frac{1}{2}}$$

$$2B-3C = 2+A$$
$$= 2+1/2$$
$$\boxed{2B-3C = 5/2}$$

$$2B-3C = 5/2$$
$$+\ 3B+3C = 3/2$$
$$5B+0 = 5/2+3/2$$

$$5B = \frac{8}{2} = 4$$
$$\therefore \boxed{B=\frac{4}{5}}$$

$$B+C = 1/2$$
$$C = 1/2 - B$$
$$C = 1/2 - 4/5$$

$$C = \frac{5-8}{10}$$

$$\boxed{C = \frac{-3}{10}}$$

eq^n (I) becomes

$$\frac{s^2+2s-3}{s(s-3)(s+2)} = \frac{1}{2} \times \frac{1}{s} + \frac{4}{5} \times \frac{1}{s-3} + \left(\frac{-3}{10}\right) \times \frac{1}{s+2}$$

Taking L^{-1} on both sides

$$L^{-1}\left[\frac{s^2+2s-3}{s(s-3)(s+2)}\right] = \frac{1}{2} L^{-1}\left[\frac{1}{s}\right] + \frac{4}{5} L^{-1}\left[\frac{1}{s-3}\right] - \frac{3}{10} L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2} \times 1 + \frac{4}{5} e^{3t} - \frac{3}{10} e^{-2t}$$

2. Find $L^{-1}\left[\frac{2s^2-1}{(s^2+1)(s^2+4)}\right]$

$\longrightarrow$

Consider

$$\frac{2s^2-1}{(s^2+1)(s^2+4)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+4)} \quad \text{— (I)}$$

$$\frac{2s^2-1}{(s^2+1)(s^2+4)} = \frac{(As+B)(s^2+4) + (Cs+D)(s^2+1)}{(s^2+1)\cdot(s^2+4)}$$

$$2s^2-1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$2s^2-1 = As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D$$

$$2s^2-1 = s^3(A+C) + s^2(B+D) + s(4A+C) + 4B + D$$

Equating powers of s.

$A+C=0$, $B+D=2$, $4A+C=0$, $4B+D=-$

$\therefore$ $\boxed{A=0}$ $\qquad$ $B+D=2$

$\boxed{C=0}$ $\qquad$ $4B+D=-1$

$\qquad\qquad -\quad -\quad +$

$$-3B = 3$$

$$\boxed{B=-1}$$

$$\therefore \; B+D=2$$

$$(-1)+D=2$$

$$\boxed{D=3}$$

eq^n (I) becomes

$$\frac{2s^2-1}{(s^2+1)(s^2+4)} = \frac{0\times s+(-1)}{(s^2+1)} + \frac{0\times s+3}{(s^2+4)}$$

$$\frac{2s^2-1}{(s^2+1)(s^2+4)} = \frac{-1}{s^2+1} + \frac{3}{s^2+4}$$

Taking L^{-1} on both sides

$$L^{-1}\left[\frac{2s^2-1}{(s^2+1)(s^2+4)}\right] = L^{-1}\left[\frac{-1}{s^2+1}\right] + L^{-1}\left[\frac{3}{s^2+4}\right]$$

$$= -L^{-1}\left[\frac{1}{s^2+1}\right] + 3L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= \frac{-1}{1}\sin t + 3\times\frac{1}{2}\sin 2t$$

$$\boxed{L^{-1}\left[\frac{2s^2-1}{(s^2+1)(s^2+4)}\right] = -\sin t + \frac{3}{2}\sin 2t}$$

3. Find $L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right]$

Sol^n:

consider

$$\frac{1}{(s+1)(s^2+2s+2)} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+2s+2)} \quad \text{--- (I)}$$

$$\frac{1}{(s+1)(s^2+2s+2)} = \frac{A(s^2+2s+2)+(Bs+C)(s+1)}{(s+1)(s^2+2s+2)}$$

$$1 = As^2+2As+2A+Bs^2+Bs+Cs+C$$

$$1 = s^2(A+B)+s(2A+B+C)+2A+C$$

Equating powers of s

$A+B=0$, $2A+B+C=0$, $2A+C=1$

$B=-A$, $2A+(-A)+C=0$

$A+C=0$

$$2A + C = 1 \qquad A + C = 0$$
$$A + C = 0 \qquad C = -A$$
$$- \quad - \quad - \qquad \boxed{C = -1}$$
$$A + 0 = 1$$
$$\boxed{A = 1}$$

$$B = -A$$
$$\therefore \boxed{B = -1}$$

eq^n (1) becomes

$$\frac{1}{(s+1)(s^2+2s+2)} = \frac{1}{s+1} + \frac{(-1)s + (-1)}{(s^2+2s+2)}$$

$$\frac{1}{(s+1)(s^2+2s+2)} = \frac{1}{(s+1)} + \frac{-s-1}{(s^2+2s+2)}$$

$$= \frac{1}{s+1} - \frac{s+1}{s^2+2s+2}$$

Taking L^{-1} on both sides

$$L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{s+1}{s^2+2s+1+1}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]$$

$$= e^{-t} - e^{-t} L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$= e^{-t} - e^{-t}\cos t.$$

$$\therefore \boxed{L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right] = e^{-t}[1-\cos t]}$$

4. Find $L^{-1}\left[\dfrac{s}{(s+1)^2(s^2+1)}\right]$

→ Consider,

$$\frac{s}{(s+1)^2(s^2+1)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{Cs+D}{(s^2+1)} \quad \text{— (I)}$$

$$s = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2$$

$$s = A(s^3+s+s^2+1) + B(s^2+1) + (Cs+D)(s^2+2s+1)$$

$$s = As^3 + As + As^2 + A + Bs^2 + B + Cs^3 + 2Cs^2 + Cs + Ds^2 + 2Ds + D$$

$$s = s^3(A+C) + s^2(A+B+2C+D) + s(A+C+2D) + A+B+D$$

Equating powers of 's'

$A+C=0$, $A+B+2C+D=0$, $A+C+2D=1$, $A+B+D=0$

$\therefore C=-A$ $\quad A+B+D+2C=0$ $\quad$ Put $C=-A$ $\quad 0+B+\frac{1}{2}=0$

$\therefore \boxed{A=0}$ $\quad 0+2C=0$ $\quad A-A+2D=1$

$\boxed{C=0}$ $\quad 2D=1$ $\quad \boxed{B=\frac{-1}{2}}$

$\boxed{D=\frac{1}{2}}$

$\therefore$ eq^n (I) becomes

$$\frac{s}{(s+1)^2(s^2+1)} = 0 + \frac{-1/2}{(s+1)^2} + \frac{1/2}{s^2+1}$$

Taking L^{-1} on both sides

$$L^{-1}\left[\frac{s}{(s+1)^2(s^2+1)}\right] = \frac{-1}{2} L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{-1}{2} e^{-t} L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{-1}{2} te^{-t} + \frac{1}{2}\sin t$$

$$\therefore \boxed{L^{-1}\left[\frac{s}{(s+1)^2(s^2+1)}\right] = \frac{1}{2}\left[\sin t - te^{-t}\right]}$$

Imp

Convolution Theorem

$$L^{-1}[\bar{F_1}(s)\bar{F_2}(s)] = \int_0^t F_1(u)F_2(t-u)\,du$$

Examples.

1. Solve by convolution theorem

$$L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]$$

$\rightarrow$

Let $\bar{F_1}(s) = \frac{1}{s^2+1}$

$\bar{F_2}(s) = \frac{s}{s^2+4}$

$$L^{-1}[\bar{F_1}(s)] = L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$F_1(t) = \sin t$$

$$\therefore \boxed{F_1(u) = \sin u}$$

$$\bar{F_2}(s) = \frac{s}{s^2+4}$$

$$L^{-1}[\bar{F_2}(s)] = L^{-1}\left[\frac{s}{s^2+4}\right]$$

$$F_2(t) = \cos 2t$$

$$\therefore F_2(t-u) = \cos 2(t-u)$$

$$\boxed{F_2(t-u) = \cos(2t-2u)}$$

by Convolution Theorem

$$L^{-1}[\bar{F}_1(s)\,\bar{F}_2(s)] = \int_0^t F_1(u)\,F_2(t-u)\,du$$

$$L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right] = \int_0^t \sin u \cdot \cos(2t-2u)\,du$$

$$= \frac{1}{2}\int_0^t 2\sin u \cdot \cos(2t-2u)\,du$$

$$2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$$

$$= \frac{1}{2}\int_0^t [\sin(u+2t-2u) + \sin(u-(2t-2u))]\,du$$

$$= \frac{1}{2}\int_0^t \sin(2t-u) + \sin(3u-2t)\,du$$

$$= \frac{1}{2}\left[\frac{-\cos(2t-u)}{(0-1)} + \left(\frac{-\cos(3u-2t)}{(3-0)}\right)\right]_0^t$$

$$= \frac{1}{2}\left[\cos(2t-t) - \frac{\cos(3t-2t)}{3} - \left(\cos(2t-0) - \frac{\cos(0-2t)}{3}\right)\right]$$

$$= \frac{1}{2}\left[\cos t - \frac{\cos t}{3} - \left[\cos 2t - \frac{\cos 2t}{3}\right]\right.$$

$$= \frac{1}{2}\left[\frac{3\cos t - \cos t}{3} - \left[\frac{3\cos 2t - \cos 2t}{3}\right]\right]$$

$$= \frac{1}{2}\left[\frac{2\cos t}{3} - \frac{2\cos 2t}{3}\right]$$

$$= \frac{2}{6}\left[\cos t - \cos 2t\right]$$

$$= \frac{1}{3}\left[\cos t - \cos 2t\right]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\left[\cos t - \cos 2t\right]$$

2 By using convalution theorem Solve $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

→

Sol^n, Let $\bar{F}_1(s) = \frac{1}{s(s+1)}$

$$\bar{F}_2(s) = \frac{1}{s+2}$$

$$L^{-1}[\bar{F}_1(s)] = L^{-1}\left[\frac{1}{s(s+1)}\right]$$

$$F_1(t) = \int_0^t e^{-t}\,dt = \left[\frac{e^{-t}}{-1}\right]_0^t$$

$$F_1(t) = 1 - e^{-t}$$
$$F_1(u) = 1 - e^{-u}$$

$$\bar{F}_2(s) = \frac{1}{s+2}$$

$$L^{-1}[\bar{F}_2(s)] = L^{-1}\left[\frac{1}{s+2}\right]$$

$$F_2(t) = e^{-2t}.$$
$$F_2(t-u) = e^{-2(t-u)}$$

$$\therefore \boxed{F_2(t-u) = e^{-2t+2u}}$$

by convolution Th^m

$$L^{-1}[\bar{F}_1(s)\,\bar{F}_2(s)] = \int_0^t F_1(u).\,F_2(t-u)\,du$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \int_0^t (1-e^{-u})\,e^{-2t+2u}\,du$$

$$= \int_0^t (1-e^{-u})\,e^{-2t}\,e^{2u}\,du$$

$$= e^{-2t}\int_0^t (1-e^{-u})\,e^{2u}\,du$$

$$= e^{-2t}\int_0^t e^{2u} - e^{2u}\,e^{-u}\,du$$

$$= e^{-2t}\int_0^t (e^{2u} - e^{u})\,du$$

$$= e^{-2t}\left[\frac{e^{2u}}{2} - e^{u}\right]_0^t$$

$$= e^{-2t}\left[\frac{e^{2t}}{2} - e^{t} - \left(\frac{e^{0}}{2} - e^{0}\right)\right]$$

$$= e^{-2t}\left[\frac{e^{2t}}{2} - e^{t} - \left(\frac{1}{2} - 1\right)\right]$$

$$= e^{-2t}\left[\frac{e^{2t}}{2} - e^{t} - \left(\frac{-1}{2}\right)\right]$$

$$= e^{-2t}\left[\frac{e^{2t}}{2} - e^{t} + \frac{1}{2}\right]$$

~~Solution of different~~

H.W.

3. $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

→ Let $\bar{F}_1(s) = \frac{s}{s^2+a^2}$, $\bar{F}_2(s) = \frac{s}{s^2+b^2}$

$L^{-1}[\bar{F}_1(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right]$ $\quad$ $L^{-1}[\bar{F}_2(s)] = L^{-1}\left[\frac{s}{s^2+b^2}\right]$

$F_1(t) = \cos at$ $\quad$ $F_2(t) = \cos bt$

$\therefore F_2(t-u) = \cos b(t-u)$

$\therefore \boxed{F_1(u) = \cos au}$ $\quad$ $\therefore \boxed{F_2(t-u) = \cos(bt-bu)}$

by convolution theorem,

$$L^{-1}[\bar{F}_1(s).\bar{F}_2(s)] = \int_0^t F_1(u).F_2(t-u)\,du$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \int_0^t \cos au \cdot \cos(bt-bu)\, du$$

$$= \frac{1}{2}\int_0^t 2\cos au \cdot \cos(bt-bu)\, du$$

$$= \frac{1}{2}\int_0^t [\cos(au+bt-bu)+\cos(au-bt+bu)]\, du$$

$$= \frac{1}{2}\left[\frac{\sin((a-b)u+bt)}{a-b} + \frac{\sin((a+b)u-bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin((a-b)t+bt)}{a-b} + \frac{\sin((a+b)t-bt)}{a+b}\right.$$

$$\left. - \left(\frac{\sin((a-b)0+bt)}{a-b} + \frac{\sin((a+b)\times 0-bt)}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b} + \frac{\sin(at+bt-bt)}{a+b}\right.$$

$$\left. - \left(\frac{\sin(0+bt)}{a-b} + \frac{\sin(0-bt)}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \left(\frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b}+\frac{1}{a+b}\right) - \sin bt\left(\frac{1}{a-b}-\frac{1}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{2a}{a^2-b^2}\right) - \sin bt\left(\frac{2b}{a^2-b^2}\right)\right]$$

$$\therefore \boxed{L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}}$$

95%

Solution of differential equation with Constant Coefficient.

$$L[F'(t)] = s\bar{F}(s) - F(0)$$

$$L[F''(t)] = s^2\bar{F}(s) - sF(0) - F'(0)$$

$$L[F'''(t)] = s^3\bar{F}(s) - s^2F(0) - sF'(0) - F''(0)$$

Examples :

1. $$\frac{d^3y}{dt^3} + \frac{2d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

$$y = 1, \quad \frac{dy}{dt} = 2, \quad \frac{d^2y}{dt^2} = 2 \quad \text{at } t = 0$$

→

Sol^n. We have

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

$$y'''(t) + 2y''(t) - y'(t) - 2y(t) = 0 \quad \text{—— (A)}$$

at $t = 0$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 2$

Taking L.T ~~on~~ ~~bo~~ to eq^n (A)

$$L[y'''(t)] + 2L[y''(t)] - L[y'(t)] - 2L[y(t)] = 0$$

$$s^3\bar{y}(s) - s^2y(0) - sy'(0) - y''(0) + 2[s^2\bar{y}(s) - sy(0) - y'(0)]$$
$$- [s\bar{y}(s) - y(0)] - 2\bar{y}(s) = 0$$

$$s^3\bar{y}(s) - s^2(1) - s(2) - 2 + 2s^2\bar{y}(s) - 2s(1) - 2(2)$$
$$- s\bar{y}(s) + 1 - 2\bar{y}(s) = 0$$

$$\bar{y}(s)[s^3 + 2s^2 - s - 2] - s^2 - 2s - 2 - 2s - 4 + 1 = 0$$

$$\bar{y}(s)(s^3 + 2s^2 - s - 2) - s^2 - 4s - 5 = 0$$

$$\bar{y}(s)[s^3 + 2s^2 - s - 2] = s^2 + 4s + 5$$

$$\bar{y}(s) = \frac{s^2 + 4s + 5}{s^3 + 2s^2 - s - 2} \longrightarrow (B)$$

Consider,

$s^3 + 2s^2 - s - 2$

$$\begin{array}{c|cccc} 1 & 1 & 2 & -1 & -2 \\ & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$$(s+1)(s^2 + s - 2) = 0$$
$$(s+1)(s+2)(s-1) = 0$$

eqⁿ (B) becomes

$$\bar{y}(s) = \frac{s^2 + 4s + 5}{s^3 + 2s^2 - s - 2} = \frac{s^2 + 4s + 5}{(s+1)(s-1)(s+2)}$$

$$= \frac{A}{(s+1)} + \frac{B}{(s-1)} + \frac{C}{(s+2)}$$

(c)

$$s^2+4s+5 = A(s-1)(s+2) + B(s+1)(s+2) + C(s-1)(s+1)$$

$$= A(s^2+2s-s-2) + B(s^2+2s+s+2) + C(s^2+s-s-1)$$

$$= A(s^2+s-2) + B(s^2+3s+2) + C(s^2-1)$$

$$= As^2 + As - 2A + Bs^2 + 3Bs + 2D + Cs^2 - C$$

$$s^2+4s+5 = s^2(A+B+C) + s(A+3B) + 2B - C - 2A$$

$\therefore$ $A+B+C=1$, $A+3B=4$, $-2A+2B-C=5$

$$\begin{array}{r} -2A+2B-C=5 \\ A+B+C=1 \\ \hline -A+3B+0=6 \end{array}$$

$$\begin{array}{r} A+3B=4 \\ -A+3B=6 \\ \hline 6B=10 \end{array}$$

$$\boxed{B=\frac{5}{3}}$$

$A+3B=4$

$A=4-3B$

$=4-3\times\frac{5}{3}$

$=4-5$

$$\boxed{A=-1}$$

$\therefore$ $A+B+C=1$

$-1+\frac{5}{3}+C=1$

$\therefore$ $C=1+1-\frac{5}{3}$

$C = 2-\frac{5}{3}$

$$C = \frac{6-5}{3}$$

$$\boxed{C = \frac{1}{3}}$$

eq^n (c) becomes

$$\frac{s^2+4s+5}{s^3+2s^2-s-2} = \frac{-1}{(s+1)} + \frac{5}{3} \times \frac{1}{s-1} + \frac{1}{3} \times \frac{1}{s+2}$$

L^{-1} on both sides

$$L^{-1}[\bar{y}(s)] = L^{-1}\left[\frac{s^2+2s+5}{s^3+2s^2-2s-2}\right] = -1\, L^{-1}\left[\frac{1}{s+1}\right] + \frac{5}{3} L^{-1}\left[\frac{1}{s-1}\right]$$

$$+ \frac{1}{3} L^{-1}\left[\frac{1}{s+2}\right]$$

$$\boxed{y(t) = -e^{-t} + \frac{5}{3} e^{t} + \frac{1}{3} e^{-2t}}$$

2. $\frac{d^2x}{dt^2} + 9x = \cos 2t$ if $x(0) = 1$, $x\left(\frac{\pi}{2}\right) = -1$

$\rightarrow$

We have,

$$\frac{d^2x}{dt^2} + 9x = \cos 2t$$

$x''(t) + 9x(t) = \cos 2t$ if $x(0) = 1$,

Taking L.T on both sides

$$\frac{-1+5}{5} \quad \frac{4}{5}$$

$$L[x''(t)] + 9L[x(t)] = L[\cos 2t]$$

$$s^2\bar{x}(s) - s\,x(0) - x'(0) + 9\bar{x}(s) = \frac{s}{s^2+4}$$

$$s^2(\bar{x}(s)) - s(1) - k + 9\bar{x}(s) = \frac{s}{s^2+4}.$$

$$\bar{x}(s)(s^2+9) - s - k = \frac{s}{s^2+4}$$

$$\bar{x}(s)(s^2+9) = \frac{s}{s^2+4} + s + k$$

$$\bar{x}(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

Taking L^{-1} on both sides.

$$L^{-1}[\bar{x}(s)] = L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] + L^{-1}\left[\frac{s}{s^2+9}\right] + L^{-1}\left[\frac{k}{s^2+9}\right]$$

$$= \underbrace{L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right]}_{A_1} + \cos 3t + \frac{k}{3}\sin 3t \qquad \text{— (A)}$$

Consider

$$A_1 = \frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{(s^2+4)} + \frac{Cs+D}{(s^2+9)} \qquad \text{— (B)}$$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{(As+B)(s^2+9) + (Cs+D)(s^2+4)}{(s^2+4)(s^2+9)}$$

$$s = As^3 + 9As + Bs^2 + 9B + Cs^3 + 4Cs + Ds^2 + 4D$$

$$s = s^3[A+C] + s^2[B+D] + s[9A+4C] + 9B+4D$$

equating powers of s

$A+C=0$, $B+D=0$, $9A+4C=1$, $9B+4D=0$

$A=-C$ $\quad$ $B=-D$ $\quad$ Put $A=-C$ $\quad$ $9(-D)+4D=0$

put $C = -\frac{1}{5}$ $\quad$ Put $D=0$ $\quad$ $9(-C)+4C=1$ $\quad$ $-9D+4D=0$

$\therefore \boxed{B=0}$ $\quad$ $-9C+4C=1$ $\quad$ $-5D=0$

$\therefore \boxed{A=\frac{1}{5}}$ $\quad$ $-5C=1$ $\quad$ $\therefore \boxed{D=0}$

$\boxed{C=-\frac{1}{5}}$

eq^n (B) becomes

$$A_1 = \frac{s}{(s^2+4)(s^2+9)} = \frac{\frac{1}{5}s+0}{s^2+4} + \frac{-\frac{1}{5}s+0}{s^2+9}$$

$$= \frac{1}{5}\frac{s}{s^2+4} - \frac{1}{5}\frac{s}{s^2+9}$$

$$= \frac{1}{5}L^{-1}\left[\frac{s}{s^2+2^2}\right] - \frac{1}{5}L^{-1}\left[\frac{s}{s^2+3^2}\right]$$

$$= \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$$

eq^n (A) becomes

$$L^{-1}[\bar{x}(s)] = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{k}{3}\sin 3t \quad \text{(A)}$$

$$x(t) = \frac{1}{5}\cos 2t + \frac{4\cos 3t}{5} + \frac{k}{3}\sin 3t \quad \text{--- (c)}$$

$x\left(\frac{\pi}{2}\right) = -1$ put $t = \frac{\pi}{2}$

$$x\left(\frac{\pi}{2}\right) = \frac{1}{5}\cos 2\times\frac{\pi}{2} + \frac{4}{5}\cos\left(\frac{3\pi}{2}\right) + \frac{k}{3}\sin\left(\frac{3\pi}{2}\right)$$

$$-1 = \frac{1}{5}(-1) + \frac{4}{5}\times 0 + \frac{k}{3}(-1)$$

$$-1 = \frac{-1}{5} - \frac{k}{3}$$

$$\frac{k}{3} = \frac{-1}{5} + 1$$

$$\boxed{\frac{k}{3} = \frac{4}{5}} \quad \therefore \boxed{k = \frac{12}{5}}$$

eq^n (c) becomes

$$\boxed{x(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t}$$

Solution of Simultaneous L.D.E

* Examples

1. Solve by L.T $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$

given that $x(0) = 1$, $y(0) = 0$

⇒ Sol^n, we have

$$\frac{dx}{dt} - y = e^t$$

$$x'(t) - y(t) = e^t \quad — (1)$$

$$L[x'(t)] - L[y(t)] = L[e^t]$$

$$s\bar{x}(s) - x(0) - \bar{y}(s) = \frac{1}{s-1}$$

$$s\bar{x}(s) - \bar{y}(s) - 1 = \frac{1}{s-1}$$

$$s\bar{x}(s) - \bar{y}(s) = \frac{1}{s-1} + 1$$

$$= \frac{1 + s - 1}{s-1}$$

$$s\bar{x}(s) - \bar{y}(s) = \frac{s}{s-1} \rightarrow (A)$$

Again consider

$$\frac{dy}{dt} + x = \sin t$$

$$y'(t) + x(t) = \sin t$$

$$L[y'(t)] + L[x(t)] = L[\sin t]$$

$$s\bar{y}(s) + y(0) + \bar{x}(s) = \frac{1}{s^2+1}$$

$$s\bar{y}(s) + \bar{x}(s) - 0 = \frac{1}{s^2+1}$$

$$s\bar{y}(s) + \bar{x}(s) = \frac{1}{s^2+1}$$

$$\bar{x}(s) + s\bar{y}(s) = \frac{1}{s^2+1} \quad \text{——} \; B$$

multiply by 's' to eq^n A

$$s^2\bar{x}(s) - s\bar{y}(s) = \frac{s^2}{s-1}$$

$$+ \quad \bar{x}(s) + s\bar{y}(s) = \frac{1}{s^2+1}$$

$$s^2\bar{x}(s) + \bar{x}(s) = \frac{s^2}{s-1} + \frac{1}{s^2+1}$$

$$\bar{x}(s)\,(s^2+1) = \frac{s^2}{s-1} + \frac{1}{s^2+1}$$

$$L^{-1}[\bar{x}(s)] = L^{-1}\left[\underbrace{\frac{s^2}{(s-1)(s^2+1)}}_{C_1}\right] + L^{-1}\left[\underbrace{\frac{1}{(s^2+1)(s^2+1)}}_{C_2}\right] \quad \text{--- (c)}$$

Consider.

$$C_1 = \frac{s^2}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$s^2 = A(s^2+1) + (Bs+C)(s-1)$$

$$= As^2 + A + Bs^2 - Bs + Cs - C$$

$$s^2 = s^2(A+B) + s(-B+C) + A - C$$

Equating powers of s

$A+B=1$, $-B+C=0$, $A-C=0$

$C=B$, $A-B=0$

$$\begin{aligned} A+B &= 1 \\ A-B &= 0 \\ \hline 2A &= 1 \end{aligned}$$

$\therefore \boxed{A = \frac{1}{2}}$

$\therefore \boxed{B = \frac{1}{2}}$ $\quad \therefore \boxed{C = \frac{1}{2}}$

$$\therefore L^{-1}\left[\frac{s^2}{(s-1)(s^2+1)}\right] = L^{-1}\left[\frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{s+1}{s^2+1}\right]$$

$$= \frac{1}{2} L^{-1}\left[\frac{1}{s-1}\right] + \frac{1}{2} L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$\boxed{L^{-1}\left[\frac{s^2}{(s-1)(s^2+1)}\right] = \frac{1}{2} e^t + \frac{1}{2} \cos t + \frac{1}{2} \sin t} \quad \text{— } C_1$$

Consider,

$$C_2 = \frac{1}{(s^2+1)(s^2+1)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+1)^2}$$

$$\frac{1}{(s^2+1)^2} = \frac{(s^2+1)(As+B) + (Cs+D)}{(s^2+1)^2}$$

$$1 = (s^2+1)(As+B) + (Cs+D)$$

$$1 = As^3 + Bs^2 + As + B + Cs + D$$

$$1 = As^3 + Bs^2 + s(A+C) + B + D$$

Equating powers of s

$\boxed{A=0}$, $\boxed{B=0}$, $A+C=0$, $B+D=1$

$0+C=0$ $\quad$ $0+D=1$

$\therefore \boxed{C=0}$ $\quad$ $\therefore \boxed{D=1}$

$$\therefore \cancel{C_2 = \frac{1}{(s^2+1)^2}} = \cancel{\frac{0+0}{(s^2+1)}} + \cancel{\frac{1}{(s^2+1)^2}}$$

$$\cancel{L^{-1}\left[\frac{1}{(s^2+1)^2}\right]}$$

Consider,

$$C_2 = \frac{1}{(s^2+1)^2}$$

let $\bar{F_1}(s) = \frac{1}{s^2+1}$, $\bar{F_2}(s) = \frac{1}{s^2+1}$

$L^{-1}[\bar{F_1}(s)] = \sin t$, $L^{-1}[\bar{F_2}(s)] = \sin t$

$f_1(t) = \sin t$, $F_2(t) = \sin t$

$\therefore \boxed{F_1(u) = \sin u}$, $\boxed{F_2(u) = \sin(t-u)}$

By convolution theorem

$$L^{-1}[\bar{F_1}(s) \cdot \bar{F_2}(s)] = \int_0^t F_1(u) \cdot F_2(t-u)\, du$$

$$= \int_0^t \sin u \sin(t-u)\, du$$

use $2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$

$$= \frac{1}{2}\int_0^t 2\sin u \cdot \sin(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \left(\cos(u-(t-u)) - \cos(u+(t-u))\right) du$$

$$= \frac{1}{2}\int_0^t \left(\cos(2u-t) - \cos(t)\right) du$$

$$= \frac{1}{2}\left[\frac{\sin(2u-t)}{2} - \cos t\,[u]\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin t}{2} - t\cos t - \left(\frac{\sin(-t)}{2} - 0\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin t}{2} - t\cos t - \left(\frac{-\sin t}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin t}{2} + \frac{\sin t}{2} - t\cos t\right]$$

$$= \frac{1}{2}\left[\frac{2\sin t}{2} - t\cos t\right]$$

$$L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = \frac{1}{2}\sin t - \frac{t}{2}\cos t \quad \text{—— } C_2$$

$\therefore$ eq^n (C) becomes.

$$\boxed{x(t) = \frac{1}{2}e^t + \frac{1}{2}\cos t + \sin t - \frac{t}{2}\cos t} \quad \text{... (D)}$$

Diff. w.r.t. 't'

$$\therefore x'(t) = \frac{1}{2}e^t - \frac{1}{2}\sin t + \cos t - \frac{1}{2}\left[-t\sin t + \cos t\right]$$

$$\boxed{x'(t) = \frac{1}{2}e^t - \frac{1}{2}\sin t + \frac{1}{2}\cos t + \frac{t}{2}\sin t} \quad \text{—— (2)}$$

put eq^n (2) in eq^n (1) we get

$$\boxed{y(t) = -\frac{1}{2}e^t - \frac{1}{2}\sin t + \frac{1}{2}\cos t + \frac{t}{2}\sin t}$$